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Yang–Baxter and Other Relations for Free-Fermion and Ising Models

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Eight-vertex, free fermion, and Ising models are formulated using a convention that emphasizes the algebra of the local transition operators that arise in the quantum inverse method. Equivalent classes of models, are investigated, with particular emphasis on the role of the star–triangle relations. Using these results, a natural and symmetrical parametrization is introduced and Yang–Baxter relations are constructed in an elementary way. The paper concludes with a consideration of duality, which links the present work to a recent paper of Baxter on the free fermion model.

KEY WORDS: Exactly integrable; Yang–Baxter; free-fermion; Ising model.

1. INTRODUCTION

The quantum inverse method (QIM) gives a systematic method for constructing the Bethe eigenstates of an integrable system in algebraic (operator) form.[1,2] For the zero-field, eight-vertex model, and the related Heisenberg XYZ chain, the method has been worked out in some detail,[3] and progress has been made toward the calculation of correlation functions using the QIM.[4,5] For an integrable lattice model the method applies whenever there is a family of commuting transfer matrices, since this is a necessary and sufficient condition for the existence of Yang–Baxter relations.[6] Such classes of models include six- and eight-vertex models, free-fermion models, and rectangular, triangular, and checkerboard Ising models.

The adaptation of the QIM to lattice statistical systems is more recent than the solution of well-known exactly soluble models such as the non-

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linear Schrödinger equation[7] and other quantum field models.[8] Undoubtedly one reason for this is that a great deal was already known for the statistical lattice models before the advent of QIM using other methods.[9] The QIM effectively gives a transformation from an interacting (nonlinear) system to a set of independent operators that create Bethe Ansatz states. Furthermore, recent work shows that the method can produce many useful transformations, which intertwine solutions to families of problems,[10] while for the nonlinear Schrödinger equation it is known that the QIM gives a type of Jordan–Wigner transformation.[11]

This paper is concerned with the formulation of eight-vertex and checkerboard Ising models in the language of the QIM, in the relationship between and among free-fermion models and Ising models, and with the construction of Yang–Baxter relations. The partition function for the free-fermion model was calculated by Fan and Wu[12] and the model was investigated in some detail by Felderhof.[13] More recently there has been renewed interest in the free-fermion model,[14,15] and the paper by Baxter[15] is based on a relationship with the checkerboard Ising model. The present paper is intended to prepare the way for an investigation of the relationship between the QIM and fermionization procedures for lattice models, in particular, in the relationship between the QIM and the Bogiolubov–Valatin transformation.[16] For the regular Ising model, the latter transformation has been exploited rather effectively by Abraham[17] to produce representations for n-point functions; it is hoped that a synthesis with the methods of the QIM will greatly extend the domain of these methods.

The plan of the paper is as follows. In Section 2 we give a general formulation of vertex models in the language of the QIM. In Section 3 we formulate the checkerboard Ising model as a (nonstandard) eight-vertex model, and investigate classes of equivalent Ising models. The results of this section may be used to demonstrate the S_4 symmetry of the free-fermion model noted by Bazhanov and Stroganov[14] and treated in an Ising formulation by Baxter.[15] In Section 4 the star–triangle relation is used to give a symmetrical parametrization, particularly in conjunction with elliptic functions. These results are used in Section 5 to construct Yang–Baxter relations. Duality of our constructs with the methods of Baxter[15] is discussed in Section 6. Some concluding comments are made in Section 7.

2. NOTATION AND REPRESENTATIONS

We commence with an eight-vertex model on a square lattice. Each edge of the lattice has an edge variable (or arrow), which can take one of two states, represented here by broken and solid lines, and each vertex has a weight associated with the various configurations of the four edges that meet there. In the language of the quantum inverse method, we will regard the horizontal edges as auxiliary variables and the vertical edges as quantum variables, and label them with Latin and Greek indices, respectively (see Fig. 1). For each vertex we define a local transition operator, which acts in the direct product of auxiliary and quantum spaces, as

$$\mathbf{L}=\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{2.1}$$

Here the w_{ij} are operators acting on the quantum variables whose matrix elements $w_{ij}(a, \beta)$ are the Boltzmann weights for the configuration i, j, α, β. Using the standard notation for the vertex weights (see Fig. 2), we may represent the w_{ij} as

$$\begin{aligned} w_{11}&=\begin{bmatrix} w_1 & 0 \\ 0 & w_3 \end{bmatrix}, \qquad w_{12}=\begin{bmatrix} 0 & w_7 \\ w_5 & 0 \end{bmatrix} \\ w_{21}&=\begin{bmatrix} 0 & w_6 \\ w_8 & 0 \end{bmatrix}, \qquad w_{22}=\begin{bmatrix} w_4 & 0 \\ 0 & w_2 \end{bmatrix} \end{aligned} \tag{2.2}$$

The local transition operators combine to form row-to-row transfer matrices represented as operators in the quantum variables for rows of N spins. Assuming periodic boundary conditions, this takes the form

$$\mathbf{T}=\mathrm{Tr}(\mathbf{L}_1\mathbf{L}_2\cdots\mathbf{L}_N) \tag{2.3}$$

where the matrix multiplication and trace operations are in the auxiliary

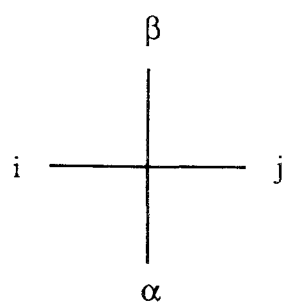

Fig. 1. Labeling of edge variables.

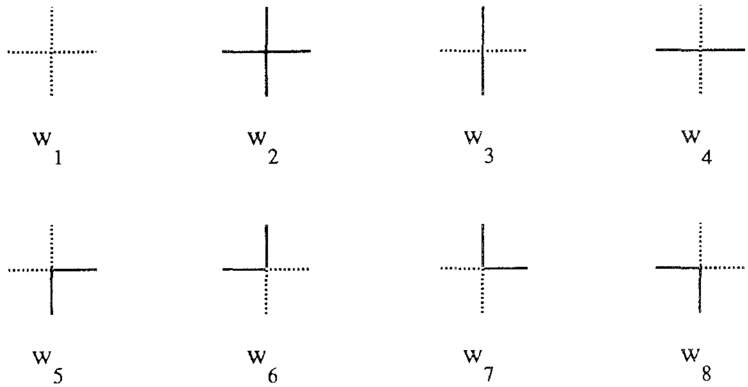

Fig. 2. Standard weights for the eight-vertex model.

variables. For a system with M rows, the partition function per site $z_{MN}(w_1,..., w_8)$ is given as a trace over the quantum variables, namely

$$z_{MN}(w_1,..., w_8) = [\mathrm{Tr}(\mathbf{T}^M)]^{1/MN} \tag{2.4}$$

We note that changing the normalization of the weights has only a trivial effect on $\mathbf{L}$, $\mathbf{T}$, and z_{MN}: multiplying the weights by a factor ρ multiplies $\mathbf{L}$ and z_{MN} by ρ and $\mathbf{T}$ by ρ^N.

We shall refer to Eq. (2.2) as the "standard form" for an eight-vertex model. Suppose now that U and V are operators each acting on one auxiliary or quantum variable, respectively. Their direct product $U \otimes V$ may be used to effect a similarity transformation on the local transition operator,

$$\mathbf{L}' = (U \otimes V)^{-1} \mathbf{L}(U \otimes V) \tag{2.5}$$

whose effect on the transfer matrix is a similarity transformation in the quantum variables only, namely

$$\mathbf{T}' = (V_1 V_2 \cdots V_N)^{-1} \mathbf{T}(V_1 V_2 \cdots V_N) \tag{2.6}$$

Here we use a rather obvious notation for the direct products of operators V_k acting on the various quantum variables along a row rather than employ the direct product notation in two different contexts. The partition function is invariant under the local transformation (2.5).

These facts lead to a more general definition of an eight-vertex model, as follows. First, for a standard model, an alternative way to write the local transition operator is in the form

$$2\mathbf{L} = u_1 \sigma_4 \otimes \sigma_4 + u_2 \sigma_3 \otimes \sigma_3 + u_3 \sigma_3 \otimes \sigma_4 + u_4 \sigma_4 \otimes \sigma_3 + u_5 \sigma_1 \otimes \sigma_1 + u_6 \sigma_2 \otimes \sigma_2 + i u_7 \sigma_2 \otimes \sigma_1 - i u_8 \sigma_1 \otimes \sigma_2 \tag{2.7}$$

where σ_4 is the unit matrix, σ_i, $i = 1, 2, 3$, are the Pauli spin matrices, and the factor 2 is inserted so that the standard weights w_i are related to the weights u_i by

$$\begin{aligned} 2w_1 &= u_1 + u_2 + u_3 + u_4, & 2w_2 &= u_1 + u_2 - u_3 - u_4 \\ 2w_3 &= u_1 - u_2 + u_3 - u_4, & 2w_4 &= u_1 - u_2 - u_3 + u_4 \\ 2w_5 &= u_5 + u_6 + u_7 + u_8, & 2w_6 &= u_5 + u_6 - u_7 - u_8 \\ 2w_7 &= u_5 - u_6 + u_7 - u_8, & 2w_8 &= u_5 - u_6 - u_7 + u_8 \end{aligned} \tag{2.8}$$

These relations are an involution between the w_i and u_i , which is related to duality in the case of the free-fermion model. Clearly, any local transition operator of the form (2.7), where now the operators σ_i are arbitrary representations of the Pauli matrix algebra, will be similar to one or more eight-vertex models in standard form for some choice(s) of U and V, since all two-dimensional representations of the Pauli algebra are equivalent. In the new basis the local transition operator, written in the form (2.1), may have all 16 weights nonzero and therefore appear as a 16-vertex model, but we shall regard it as an eight-vertex model nevertheless.

We conclude this section by investigating the most general similarity transformations connecting two eight-vertex models that are both in standard form. We will distinguish the σ operators after the similarity transformation with a prime. Clearly, the form (2.7) can only be preserved if σ_3 and σ'_3 (in both auxiliary and quantum spaces) are diagonal, since they are coupled with σ_4, which is invariant. Thus, the possible similarity transformations are

$$\begin{aligned} \sigma'_1 &= U \sigma_1 U^{-1} = (\mu \operatorname{ch} N)\, \sigma_1 + (i \operatorname{sh} N)\, \sigma_2 \\ \sigma'_2 &= U \sigma_2 U^{-1} = (-i\mu \operatorname{sh} N)\, \sigma_1 + (\operatorname{ch} N)\, \sigma_2 \\ \sigma'_3 &= U \sigma_3 U^{-1} = \mu \sigma_3, \qquad \mu = \pm 1 \end{aligned} \tag{2.9}$$

with similar relations for $V\sigma_i V^{-1}$. (We use the compact notation sh and ch for the hyperbolic sine and cosine functions.) Using the subscripts a and q on the parameters μ and N to denote whether they pertain to U or V, there are four free parameters, in terms of which the effect of the transformation on the weights u_i are

$$\begin{aligned}
&u_1' = u_1, \qquad u_2' = \mu_a\mu_q u_2, \qquad u_3' = \mu_a u_3, \qquad u_4' = \mu_q u_4 \\
&u_5' = \mu_a\mu_q(\text{ch } N_a \text{ ch } N_q u_5 - \text{sh } N_a \text{ sh } N_q u_6 \\
&\qquad + \text{sh } N_a \text{ ch } N_q u_7 - \text{ch } N_a \text{ sh } N_q u_8) \\
&u_6' = (-\text{sh } N_a \text{ sh } N_q u_5 + \text{ch } N_a \text{ ch } N_q u_6 \\
&\qquad - \text{ch } N_a \text{ sh } N_q u_7 + \text{sh } N_a \text{ ch } N_q u_8) \\
&u_7' = \mu_q(\text{sh } N_a \text{ ch } N_q u_5 - \text{ch } N_a \text{ sh } N_q u_6 \\
&\qquad + \text{ch } N_a \text{ ch } N_q u_7 - \text{sh } N_a \text{ sh } N_q u_8) \\
&u_8' = \mu_a(-\text{ch } N_a \text{ sh } N_q u_5 + \text{sh } N_a \text{ ch } N_q u_6 \\
&\qquad - \text{sh } N_a \text{ sh } N_q u_7 + \text{ch } N_a \text{ ch } N_q u_8)
\end{aligned} \tag{2.10}$$

It is apparent that μ_a and μ_q bring about permutations of the weights w_i, while N_a and N_q affect the symmetry in the weights w_5 to w_8, so we look at their effect independently. With N_a and N_q equal to zero, the transformations (2.10) are equivalent to reversing all of the auxiliary and/or quantum variables: the permutations of the weights that this induces are shown in Table I. When μ_a and μ_q are set to $+1$, Eqs. (2.10) are equivalent to

$$\begin{aligned}
w_5' &= \exp(N_a - N_q)\, w_5 \\
w_6' &= \exp(-N_a + N_q)\, w_6 \\
w_7' &= \exp(N_a + N_q)\, w_7 \\
w_8' &= \exp(-N_a - N_q)\, w_8
\end{aligned} \tag{2.11}$$

which enables any eight-vertex model to be transformed into a representation where $w_5 = w_6$ and $w_7 = w_8$.

Table I. Permutations Under Similarity Transformation

w_1'	w_2'	w_3'	w_4'	w_5'	w_7'
w_1	w_2	w_3	w_4	$\pm w_5$	$\pm w_7$
w_2	w_1	w_4	w_3	$\pm w_5$	$\pm w_7$
w_3	w_4	w_1	w_2	$\pm w_7$	$\pm w_5$
w_4	w_3	w_2	w_1	$\pm w_7$	$\pm w_5$

3. CHECKERBOARD ISING MODEL

Checkerboard Ising models are readily formulated in the notation of Section 2, and their relationship to free-fermion eight-vertex models becomes rather transparent. Given an eight-vertex lattice, shown by broken lines in Fig. 3, we set the edge variables in one-to-one correspondence with Ising spins, that is, each two-state edge variable is replaced by a two-state spin. We then join these spins by bonds to form a diagonal checkerboard Ising model. Alternate faces of the Ising model contain vertices of the original eight-vertex model, and this creates the checkerboard. In Fig. 3, and throughout this paper, bonds are normalized by kT. The local transition operators now correspond to a face of an Ising model and the alternate diagonal rows of spins are auxiliary and quantum variables, indicated in Fig. 3 as solid and hollow circles, respectively. From this figure it is seen that, in the spin variables, the operators w_{ij} may be written as

$$
\begin{aligned}
w_{11} &= \begin{bmatrix} \exp(J_1+J_2+J_3+J_4) & \exp(J_1+J_2-J_3-J_4) \\ \exp(-J_1-J_2+J_3+J_4) & \exp(-J_1-J_2-J_3-J_4) \end{bmatrix} \\
w_{12} &= \begin{bmatrix} \exp(J_1-J_2-J_3+J_4) & \exp(J_1-J_2+J_3-J_4) \\ \exp(-J_1+J_2-J_3+J_4) & \exp(-J_1+J_2+J_3-J_4) \end{bmatrix} \\
w_{21} &= \begin{bmatrix} \exp(-J_1+J_2+J_3-J_4) & \exp(-J_1+J_2-J_3+J_4) \\ \exp(J_1-J_2+J_3-J_4) & \exp(J_1-J_2-J_3+J_4) \end{bmatrix} \\
w_{22} &= \begin{bmatrix} \exp(-J_1-J_2-J_3-J_4) & \exp(-J_1-J_2+J_3+J_4) \\ \exp(J_1+J_2-J_3-J_4) & \exp(J_1+J_2+J_3+J_4) \end{bmatrix}
\end{aligned}
\tag{3.1}
$$

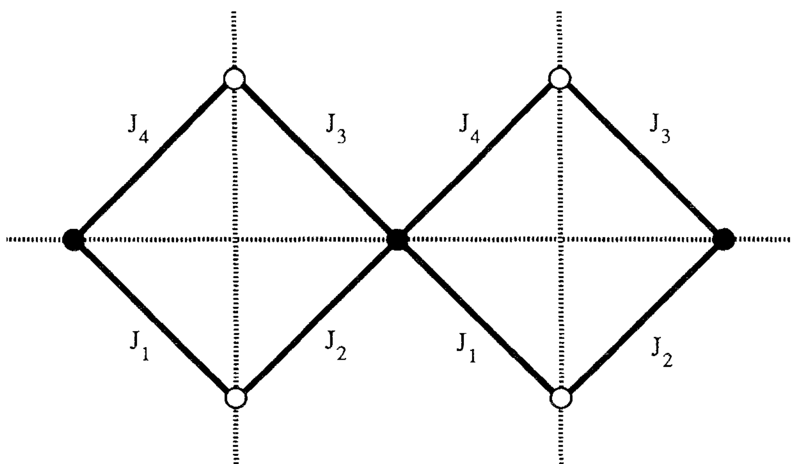

Fig. 3. Ising model (circles and solid lines) on eight-vertex edge variables (broken lines).

To see that this is equivalent to an eight-vertex model, and to calculate the weights in standard form, these w_{ij} may be written in terms of the standard operators σ_i followed by the similarity transformation $\sigma_1' = -\sigma_3$, $\sigma_3' = \sigma_1$ in both auxiliary and quantum spaces. This brings the model to standard form (2.7) with the weights

$$\begin{aligned} u_1 &= 2\,\mathrm{ch}(J_1 + J_2 + J_3 + J_4), & u_2 &= 2\,\mathrm{ch}(J_1 - J_2 + J_3 - J_4) \\ u_3 &= 2\,\mathrm{ch}(J_1 - J_2 - J_3 + J_4), & u_4 &= 2\,\mathrm{ch}(J_1 + J_2 - J_3 - J_4) \\ u_5 &= 2\,\mathrm{sh}(J_1 + J_2 + J_3 + J_4), & u_6 &= 2\,\mathrm{sh}(-J_1 + J_2 - J_3 + J_4) \\ u_7 &= 2\,\mathrm{sh}(-J_1 + J_2 + J_3 - J_4), & u_8 &= 2\,\mathrm{sh}(J_1 + J_2 - J_3 - J_4) \end{aligned} \tag{3.2}$$

It is useful to introduce dual variables in the usual manner

$$\mathrm{sh}(2J_i)\,\mathrm{sh}(2\hat{J}_i) = 1, \qquad \tanh(2J_i)\,\mathrm{ch}(2\hat{J}_i) = 1$$

Substituting (3.2) into (2.8) and transforming to dual variables gives

$$\begin{aligned} w_1 &= 2P\,\mathrm{ch}(\hat{J}_1 + \hat{J}_2 + \hat{J}_3 + \hat{J}_4), & w_2 &= 2P\,\mathrm{ch}(\hat{J}_1 - \hat{J}_2 + \hat{J}_3 - \hat{J}_4) \\ w_3 &= 2P\,\mathrm{ch}(\hat{J}_1 - \hat{J}_2 - \hat{J}_3 + \hat{J}_4), & w_4 &= 2P\,\mathrm{ch}(\hat{J}_1 + \hat{J}_2 - \hat{J}_3 - \hat{J}_4) \\ w_5 &= 2P\,\mathrm{ch}(\hat{J}_1 - \hat{J}_2 + \hat{J}_3 + \hat{J}_4), & w_6 &= 2P\,\mathrm{ch}(\hat{J}_1 + \hat{J}_2 + \hat{J}_3 - \hat{J}_4) \\ w_7 &= 2P\,\mathrm{ch}(\hat{J}_1 + \hat{J}_2 - \hat{J}_3 + \hat{J}_4), & w_8 &= 2P\,\mathrm{ch}(-\hat{J}_1 + \hat{J}_2 + \hat{J}_3 + \hat{J}_4) \end{aligned} \tag{3.4}$$

where

$$P = \prod_{i=1}^{4} (\mathrm{sh}\,2J_i)^{1/2} \tag{3.5}$$

Equations (3.4) have an identical structure to Eqs. (2.5) of Ref. 15: the duality with those equations is considered in Section 6. The weights (3.4) satisfy the free-fermion condition

$$w_1 w_2 + w_3 w_4 = w_5 w_6 + w_7 w_8 \tag{3.6}$$

as a consequence of elementary properties of the hyperbolic functions. An additional formula we will need is

$$16P^2 = w_5 w_6 w_7 w_8 - w_1 w_2 w_3 w_4 \tag{3.7}$$

The right-hand side of (3.7) is invariant under similarity transformations, while the left-hand side is a function of the J_i, therefore, we regard local transition operators to be equivalent when they are related by

$$\rho(J_i')\,\mathbf{L}(J_i') = \rho(J_i)(U \otimes V)^{-1}\,\mathbf{L}(J_i)(U \otimes V) \tag{3.8}$$

where

$$\rho(J_i) = \prod_{i=1}^{4} (2\,\mathrm{sh}\,2J_i)^{-1/4} \tag{3.9}$$

that is, the similarity transformation is applied to transition operators that have been normalized by a factor $(2\,\mathrm{sh}\,2J_i)^{-1/4}$ for each Ising bond J_i. Solutions of (3.8) are most easily found by representing similarity transformations as extra Ising bonds inserted in the rows and/or columns. For example, inserting a horizontal bond H between two local transition operators is equivalent to inserting the operator

$$\exp(H)\,\sigma_4 + \exp(-H)\,\sigma_3 \tag{3.10}$$

in the auxiliary space product, remembering that we are using a representation of the Pauli matrices where σ_1 is diagonal and σ_3 has the usual (standard) form for σ_1. The inverse is given as

$$(2i\,\mathrm{sh}\,2H)^{-1}\,[\exp(H + i\pi/2)\,\sigma_4 + \exp(-H - i\pi/2)\,\sigma_3] \tag{3.11}$$

which is another bond together with a further normalization factor. The effect of this transformation on the σ_i is readily calculated to be the same as Eq. (2.9) with N replaced by $2\hat{H}$ and $\mu = 1$. We therefore write

$$\begin{aligned} U(2\hat{H}) &= \exp(H)\,\sigma_4 + \exp(-H)\,\sigma_3 \\ U^{-1}(2\hat{H}) &= (2i\,\mathrm{sh}\,2H)^{-1}\,[\exp(H + i\pi/2)\,\sigma_4 + \exp(-H - i\pi/2)\,\sigma_3] \end{aligned} \tag{3.12}$$

Similarity transformations may now be constructed as a sequence of star–triangle transformations (Fig. 4). The fundamental equations are found in Ref. 9, namely

$$\mathrm{sh}(2J_i)\,\mathrm{sh}(2L_i) = \Omega^{-1} \tag{3.13}$$

$$\mathrm{ch}(2J_i) = \mathrm{ch}(2L_j)\,\mathrm{ch}(2L_k) + \coth(2L_i)\,\mathrm{sh}(2L_j)\,\mathrm{sh}(2L_k) \tag{3.14}$$

Fig. 4. Star–triangle transformation.

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and we have written the constant on the right-hand side of (3.13) as Ω^{-1} to conform with the notation of Ref. 15. We will also follow that paper by referring to the bonds related by (3.13) as "supplements." Another form of Eq. (3.14), which may be obtained from it by straightforward algebra, is

$$\operatorname{ch}(2J_i) = -\operatorname{ch}(2J_j)\operatorname{ch}(2J_k) + \coth(2L_i)\operatorname{sh}(2J_j)\operatorname{sh}(2J_k) \tag{3.15}$$

The similarity transformation in quantum space shown in Fig. 5 now leads to the conditions

$$\begin{aligned} \operatorname{ch} 2H &= \operatorname{ch} 2J_1 \operatorname{ch} 2J_2 + \coth 2G \operatorname{sh} 2J_1 \operatorname{sh} 2J_2 \\ \operatorname{ch} 2H &= -\operatorname{ch} 2J_3 \operatorname{ch} 2J_4 + \coth 2G \operatorname{sh} 2J_3 \operatorname{sh} 2J_4 \end{aligned} \tag{3.16}$$

Expressing G and H in terms of J_i gives

$$\begin{aligned} \tanh 2G &= -(\operatorname{sh} 2J_1 \operatorname{sh} 2J_2 - \operatorname{sh} 2J_3 \operatorname{sh} 2J_4) \\ &\quad \times (\operatorname{ch} 2J_1 \operatorname{ch} 2J_2 + \operatorname{ch} 2J_3 \operatorname{ch} 2J_4)^{-1} \\ \operatorname{sech} 2H &= (\operatorname{sh} 2\hat{J}_1 \operatorname{sh} 2\hat{J}_2 - \operatorname{sh} 2\hat{J}_3 \operatorname{sh} 2\hat{J}_4 \\ &\quad \times (\operatorname{ch} 2\hat{J}_1 \operatorname{ch} 2\hat{J}_2 + \operatorname{ch} 2\hat{J}_3 \operatorname{ch} 2\hat{J}_4)^{-1} \end{aligned} \tag{3.17}$$

and from these we obtain an expression for Ω that is completely symmetrical in the J_i, namely

$$\begin{aligned} \Omega &= (\operatorname{sh} 2G \operatorname{sh} 2H)^{-1} \\ &= \prod_{i=1}^{4} \left[\frac{\operatorname{sh} 2J_i \operatorname{ch}(\hat{J}_1 + \hat{J}_2 + \hat{J}_3 + \hat{J}_4 - 2\hat{J}_i)}{\operatorname{sh} 2\hat{J}_i \operatorname{ch}(J_1 + J_2 + J_3 + J_4 - 2J_i)} \right]^{1/2} \end{aligned} \tag{3.18}$$

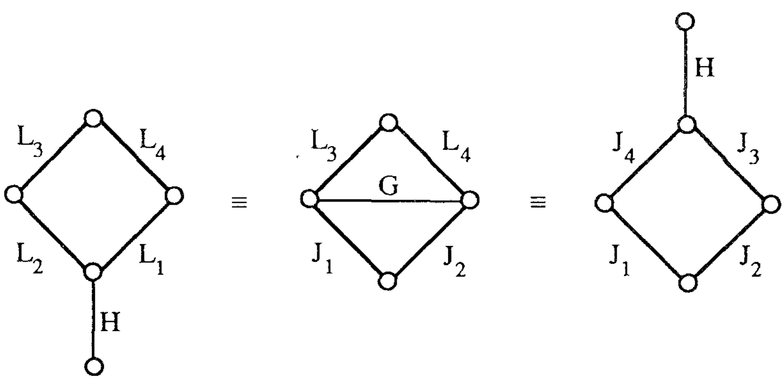

Fig. 5. Similarity transformation in quantum space.

The equations that come from transformations in auxiliary space (Fig. 6) are similar. Corresponding to (3.16), we now have

$$\begin{aligned} \operatorname{ch} 2H' &= \operatorname{ch} 2J_1 \operatorname{ch} 2J_4 + \coth 2G' \operatorname{sh} 2J_1 \operatorname{sh} 2J_4 \\ \operatorname{ch} 2H' &= -\operatorname{ch} 2J_2 \operatorname{ch} 2J_3 + \coth 2G' \operatorname{sh} 2J_2 \operatorname{sh} 2J_3 \end{aligned} \tag{3.19}$$

which may be solved for G' and H' as in (3.17). When the product $\operatorname{sh} 2G' \operatorname{sh} 2H'$ is calculated, the same symmetric expression is obtained as in Eq. (3.18).

Given a (symmetric) eight-vertex model with weights w_i, the inverse problem is to find a checkerboard Ising model equivalent to it under the relation

$$\rho_{8V} \mathbf{L}_{8V} = \rho_{IS} (U \otimes V)^{-1} \mathbf{L}_{IS} (U \otimes V) \tag{3.20}$$

We know from (3.7) that

$$\rho_{8V} = (w_5 w_6 w_7 w_8 - w_1 w_2 w_3 w_4)^{-1/4} \tag{3.21}$$

while ρ_{IS} is given by (3.9); also, there will be four solutions for $\mathbf{L}_{IS}$ if there are any. The weights given in Eqs. (3.2) or (3.4) are for a nonsymmetric eight-vertex model, so we use Eq. (2.11) and impose the conditions that the transformed weights w'_5, w'_6, w'_7, and w'_8, when substituted into (3.2), satisfy the relations $\operatorname{ch}^2 A - \operatorname{sh}^2 A = \operatorname{ch}^2 B - \operatorname{sh}^2 B$. This gives three independent conditions, two equations for N_a and N_q

$$\begin{aligned} \operatorname{ch}(2N_q) &= \Gamma^{-1} = (w_1 w_4 + w_2 w_3)/2 w_5 w_7 \\ \operatorname{ch}(2N_a) &= (\Gamma')^{-1} = (w_1 w_3 + w_2 w_4)/2 w_5 w_7 \end{aligned} \tag{3.22}$$

and the free-fermion condition, which is therefore a necessary and sufficient condition for the equivalence. The constants Γ and Γ' appearing in (3.22) are standard for the free-fermion model[18] and will be used in Section 4. For each of the four solution pairs N_a, N_q the bonds are readily deter-

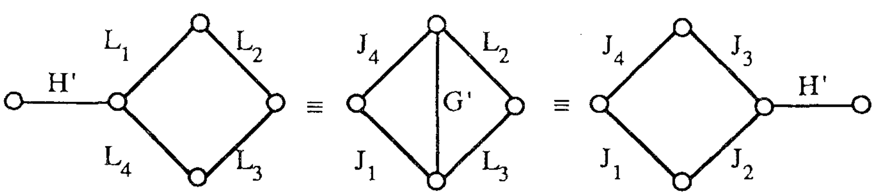

Fig. 6. Similarity transformation in auxiliary space.

mined via inverse tanh functions using ratios of equations in (3.2). These four sets of bonds will be related as follows:

$$\begin{aligned} &\text{(i)} \quad J_1, J_2, J_3, J_4 \\ &\text{(ii)} \quad L_4, L_3, L_2, L_1 \\ &\text{(iii)} \quad L_2, L_1, L_4, L_3 \\ &\text{(iv)} \quad J_3, J_4, J_1, J_2 \end{aligned} \tag{3.23}$$

Because there is a one-to-one mapping between edge variables and spins, periodic boundary conditions are equivalent for equivalent eight-vertex and Ising models, so that the normalized partition functions are the same for all finite M and N, that is,

$$\rho_{8V}(w_i)\, z_{MN}(w_i) = \rho_{IS}(J_i)\, z_{MN}(J_i) \tag{3.24}$$

Baxter[15] has shown how the equivalent sets of solutions in (3.23) may be used to "shuffle" the rows and columns so as to factor the partition function into the product of four regular Ising partition functions each with bonds J_i, L_i in the two directions—the "hidden symmetry" of Bazhanov and Stroganov.[14] It is an important ingredient of this factorization that the normalization factors that occur, (3.9), are themselves factorized.

4. ELLIPTIC FUNCTION PARAMETRIZATION

The foregoing relations lead to a natural parametrization of the J_i, based on the systematic use of the star–triangle relation. For example, if the pair of constants Ω and H is regarded as given, then J_2 and J_4 may be determined from J_1 and J_3 using Eqs. (3.16). Three such pairs of relations, involving Ω and three H's, will serve to parametrize all four J_i's. Therefore, for cyclic permutation (i, j, k) of the integers (2, 3, 4), we define the pairs of constants

$$\Gamma_i = 2(w_5 w_6 w_7 w_8)^{1/2}/(w_1 w_i + w_j w_k) \tag{4.1}$$

$$\begin{aligned} h_i &= (w_j^2 + w_k^2 - w_1^2 - w_i^2)/2(w_1 w_i + w_j w_k) \\ &= -(u_1 u_i + u_j u_k)/(w_1 w_i + w_j w_k) \end{aligned} \tag{4.2}$$

Note that in order to achieve a symmetric notation in what follows, we have not adopted the notation of Ref. 15. The terms Γ_4 and h_4 correspond to the usual Γ and h,[18] and it is well known that free-fermion models with the same value of Γ and h have commuting row-to-row transfer matrices.[18] The term Γ_3 appears in Eq. (3.22) as Γ'; models with the same values of Γ_3 and h_3 have commuting column-to-column transfer matrices. We set $G_4 = G$, $H_4 = H$, $G_3 = G'$, and $H_3 = H'$ and we define constants G_2 and H_2 by

$$\text{sh } 2G_2 \text{ sh } 2H_2 = \Omega^{-1} \tag{4.3}$$

and

$$\begin{aligned} \text{ch } 2H_2 &= \text{ch } 2J_1 \text{ ch } 2J_3 + \coth 2G_2 \text{ sh } 2J_1 \text{ sh } 2J_3 \\ \text{ch } 2H_2 &= -\text{ch } 2J_2 \text{ ch } 2J_4 + \coth 2G_2 \text{ sh } 2J_2 \text{ sh } 2J_4 \end{aligned} \tag{4.4}$$

Straightforward but tedious calculation now shows that the relationship of Γ_i and h_i to the bonds G_i and H_i is

$$\Gamma_i = \tanh 2H_i \tag{4.5}$$

$$h_i = -\coth 2G_i \text{ sech } 2H_i \tag{4.6}$$

Further, we have

$$\Omega^2 = (\Gamma_i^2 + h_i^2 - 1)/\Gamma_i^2, \qquad i = 2, 3, 4 \tag{4.7}$$

Consequently, we may employ the parameters Ω, H_2, H_3, and H_4 in place of $J_1, ..., J_4$ and this is most conveniently done using elliptic functions.

We use the standard notation for elliptic functions found, for example, in Ref. 19. For a free-fermion model with real weights, Ω^2 is real, and in order to keep the elliptic function moduli k, k' real and in the range $0 < k, k' < 1$, we distinguish three regimes as in Ref. 15:

I	$-\Omega^2 > 1$	(low temperature)
II	$0 < \Omega^2 < 1$	(high temperature)
III	$\Omega^2 < 0$	(high temperature)

The critical point is $\Omega = 1$, separating regime I from regime II. From (4.7) this is equivalent to $h_i = \pm 1$, $i = 2, 3, 4$; these are the conditions for self-duality (Section 6). In Table II we give the relation between Ω and k, k' in

Table II. Elliptic Function Definitions

Regime I	Regime II	Regime III
$\Omega > 1$	$0 < \Omega < 1$	$\Omega^2 < 0$
$k' = \Omega^{-1}$	$k' = \Omega$	$ik'/k = \Omega$
$\lambda_1 = K$	$\lambda_1 = K$	$\lambda_1 = K - iK'$
$\text{sh } 2J_i = \text{cs}(\alpha_i)$	$\text{sh } 2J_i = \text{cs}(\alpha_i)/k'$	$\text{sh } 2J_i = -i\, ds(\alpha_i)/k'$
$\text{ch } 2J_i = \text{ns}(\alpha_i)$	$\text{ch } 2J_i = ds(\alpha_i)/k'$	$\text{ch } 2J_i = -i \text{ cs}(\alpha_i)/k'$

each of the regimes, and define a parameter λ_1 in terms of the complete elliptic integrals K, K'. In place of the J_i and L_i we introduce arguments α_i by

$$J_i = J(\alpha_i), \qquad L_i = J(\lambda_1 - \alpha_i) \tag{4.8}$$

where the function $J(\alpha)$ is defined in Table II for each regime, so that J_i and L_i are automatically supplements. Similarly, we introduce three arguments λ_i in place of G_i and H_i, $i = 2, 3, 4$, by

$$H_i = J(\lambda_i), \qquad G_i = J(\lambda_1 - \lambda_i) \tag{4.9}$$

The underlying relations are the star–triangle relations (3.14) and (3.15), and these are satisfied by the bonds (4.8) provided the arguments satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 = \lambda_1 \tag{4.10}$$

The parametrization implicit in (3.16), (3.19), and (4.4) now simplies (modulo the periodicity of elliptic functions) to

$$\lambda_1 + \lambda_i = \alpha_1 + \alpha_j, \qquad \lambda_1 - \lambda_i = \alpha_i + \alpha_k \tag{4.11}$$

where (i, j, k) is a cyclic permutation of (2, 3, 4). The relationship between the α_i and λ_i turns out to be

$$\begin{aligned} 2\alpha_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \qquad & 2\alpha_2 &= \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 \\ 2\alpha_3 &= \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4, \qquad & 2\alpha_4 &= \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \end{aligned} \tag{4.12}$$

This guarantees that the arguments α_i satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\lambda_1 \tag{4.13}$$

The advantage of these arguments is that λ_1 is a function only of the fundamental constant Ω and the three λ_i each depend only on one pair Γ_i, h_i. Furthermore, row-to-row transfer matrices commute for families having the same values of λ_1 and λ_4, and column-to-column transfer matrices commute for families having the same values of λ_1 and λ_3.

5. YANG–BAXTER RELATIONS

In the QIM Yang–Baxter relations are conventionally written as[20]

$$(\mathbf{L} \otimes \mathbf{L}')\, S'' = S''(\mathbf{L}' \otimes \mathbf{L}) \tag{5.1}$$

where $\mathbf{L}$ and $\mathbf{L}'$ are local transition operators, the direct product is in auxiliary space, and the matrix R'' operates in this direct product space only. Since all the terms in this direct product involve products of quantum operators, (5.1) is a set of commutation relations. They have a simple visualization, which is easily appreciated when the three objects are set on an equal footing. Introduce the notation that to each of the local transition operators there corresponds a 4×4 matrix R according to

$$R^{\beta j}_{\alpha i} = w_{ij}(\alpha, \beta) = S''^{j\beta}_{\alpha i} \tag{5.2}$$

where the operators w_{ij} are defined in Section 2; then the Yang–Baxter relations appear as

$$R^{\beta j_1}_{\alpha i_1} R'^{\gamma j_2}_{\beta i_2} R''^{k_1 k_2}_{j_1 j_2} = R''^{j_1 j_2}_{i_1 i_2} R'^{\beta k_2}_{\alpha j_2} R^{\gamma k_1}_{\beta j_1} \tag{5.3}$$

The graphical interpretation of this as a factorizable S-matrix is shown in Fig. 7. This also serves to show the symmetry of the relations between the three local transition operators. Equation (5.1) is only one interpretation of (5.3), in which the vertical edges are taken as the quantum variables and the horizontal edges as auxiliary variables. Two other interpretations, obtained by rotating Fig. 7 through $\pm 120°$, are concerned with the commutation of either $\mathbf{L}$ or $\mathbf{L}'$ with a third operator $\mathbf{L}''$, and with different choices of auxiliary and quantum variables. In this case the Yang–Baxter relation must be written as

$$(\mathbf{L} \otimes \mathbf{L}'')\, S' = S'(\mathbf{L}'' \otimes \mathbf{L}) \tag{5.4a}$$

$$(\mathbf{L}' \otimes \mathbf{L}'')\, S = S(\mathbf{L}'' \otimes \mathbf{L}') \tag{5.4b}$$

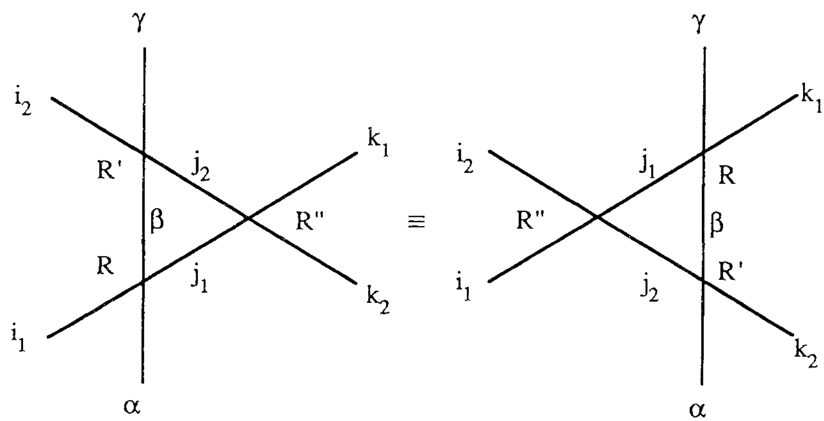

Fig. 7. Graphical interpretation of Yang–Baxter relations.

where the appropriate forms for $\mathbf{L}''$ corresponding to (5.4a) and (5.4b) are

$$R''^{k_1k_2}_{j_1j_2} = w''_{j_1k_1}(k_1, j_2) \tag{5.5a}$$

$$R''^{k_1k_2}_{j_1j_2} = w''_{j_2k_2}(j_1, k_1) \tag{5.5b}$$

In the free-fermion case, row-to-row transfer matrices commute if and only if they have the same values for Γ_4 and h_4; similarly, column-to-column transfer matrices commute if and only if they have the same values of Γ_3 and h_3. Hence, with either of the interpretations (5.5), the values of Γ_3, h_3, Γ_4, and h_4 for the third eight-vertex model w'' may be read off from the diagram because these constants "propagate along the lines," while all three operators have the same value of Ω.

The results of Section 3 make the construction of Yang–Baxter relations relatively trivial, and the interpretation of the third model $\mathbf{L}''$ quite easy. In addition to the transformation shown in Fig. 5, we need the transformation shown in Fig. 8, called a "star–star" transformation in Ref. 13. When the edges and vertices in Fig. 7 are replaced by the Ising spins and bonds of Section 3, the Yang–Baxter relations for the checkerboard Ising model are seen to be precisely the sequence of transformations shown in Fig. 9. First the bond J''_3 is moved through, using the star–star transformation, then the bonds J''_2 and J''_4 are moved through as similarity transformations, and then J''_1 induces a second star–star transformation. The condition for all this to work is that, for each of the intermediate steps, the four bonds involved must have the same Ω value. This will be so if and only if $\mathbf{L}$ and $\mathbf{L}'$ have the same values of Ω *and* H_1. There are two star–triangle conditions for each of the four moves (J''_1,..., J''_4) in Fig. 9 and these are most readily seen in terms of elliptic functions. In the order corresponding to J''_i they are

$$\alpha_2 + \alpha'_3 + \alpha''_1 = 2\lambda_1, \qquad \alpha_1 + \alpha'_4 - \alpha''_1 = 0 \tag{5.6}$$

$$\alpha_1 - \alpha'_1 + \alpha''_2 = 0, \qquad \alpha_2 - \alpha'_2 - \alpha''_2 = 0 \tag{5.7}$$

$$\alpha_4 + \alpha'_1 + \alpha''_3 = 2\lambda_1, \qquad \alpha_3 - \alpha'_2 - \alpha''_3 = 0 \tag{5.8}$$

$$\alpha_3 - \alpha'_3 + \alpha''_4 = 0, \qquad \alpha_4 - \alpha'_4 - \alpha''_4 = 0 \tag{5.9}$$

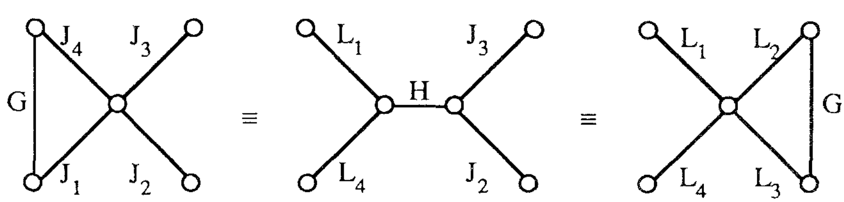

Fig. 8. Star–star transformation.

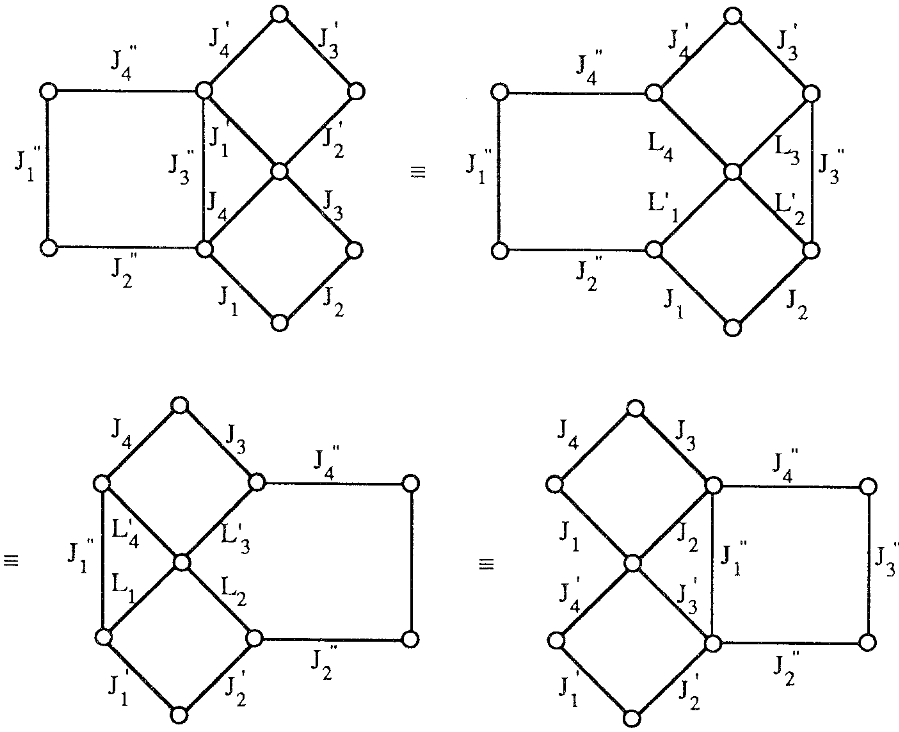

Fig. 9. Yang–Baxter relation for Ising model.

Each pair of equations implies that

$$\lambda_4 = \lambda_4' \tag{5.10}$$

i.e., $H_4 = H_4'$. In addition, they give

$$\lambda_2'' = \lambda_1 + \lambda_2 - \lambda_2' \tag{5.11}$$

$$\lambda_3'' = \lambda_3 \tag{5.12}$$

$$\lambda_4'' = \lambda_3' \tag{5.13}$$

On first inspection, this appears not to have the expected symmetry between J_i, J_i', and J_i'', but this is merely due to the unsymmetrical manner in which they appear in Fig. 9. However, the labeling there is the appropriate choice for the QIM, so we will not discuss more symmetrical labeling schemes, except to note that if such a scheme is employed (say with J_1, J_1', and J_1'' as the sides of the inner triangle on the left-hand side), then (5.11) becomes $\lambda_2 + \lambda_2' + \lambda_2'' = \lambda_1$. Since the other λ_i are simply permuted in some simple manner, this shows that the Yang–Baxter relation for the checkerboard Ising model (and the free-fermion model also) is simply a generalized star–triangle relation.

6. DUALITY RELATIONS

It is evident that our work is closely related to Baxter's.[15] In fact, it is connected by a duality transformation and to explore this we have drawn our Ising lattice and its dual in Fig. 10. The Ising lattice of Section 3 is shown in light lines and the dual lattice in heavy lines: the spins on the dual lattice are shown as squares and the bonds on the dual lattice are indicated with primes. The underlying eight-vertex lattice is shown by broken lines. Baxter constructs his relationship between an eight-vertex model and an Ising model by a mapping that is more complex than ours. The Ising spins are not in a one-to-one relationship with edge variables; rather, half of them are at vertices (solid squares) and the other half are at face centers (open squares). If the former (vertex) spins are summed out, we have an interaction around a face (IRF) model, where each face has a weight determined from the Ising bonds J_i'. The configurations of the remaining (face) spins may be set in a two-to-one relation with vertex configurations by the usual method of choosing one of the two edge states to separate unlike spins and the other edge state to separate like spins. In

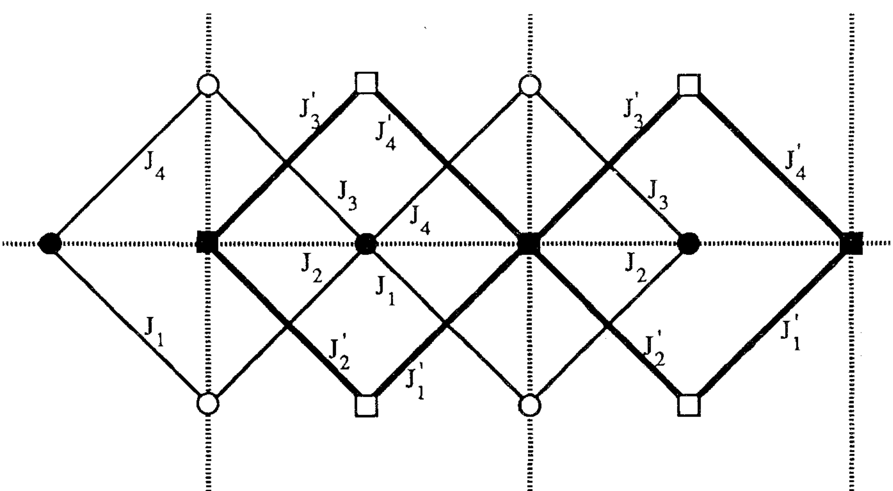

Fig. 10. Ising model (light lines) and dual Ising model (heavy lines) on eight-vertex lattice (broken lines).

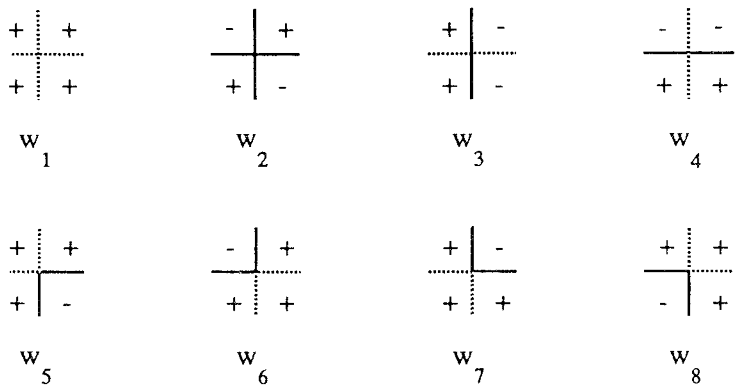

Fig. 11. Eight (of 16) face spin configurations and the corresponding eight-vertex weights.

Fig. 11 the unbroken lines separate unlike face spins; only half of the 16 spin configurations are shown, corresponding to the eight vertex configurations: the other eight spin configurations are obtained by reversing all face spins. Each eight-vertex weight is now the sum of two Boltzmann factors from the dual lattice Ising model, and an elementary calculation shows that they are given by

$$\begin{aligned} w_1' &= 2\,\mathrm{ch}(J_1' + J_2' + J_3' + J_4'), & w_2' &= 2\,\mathrm{ch}(J_1' - J_2' + J_3' - J_4') \\ w_3' &= 2\,\mathrm{ch}(J_1' - J_2' - J_3' + J_4'), & w_4' &= 2\,\mathrm{ch}(J_1' + J_2' - J_3' - J_4') \\ w_5' &= 2\,\mathrm{ch}(J_1' - J_2' + J_3' + J_4'), & w_6' &= 2\,\mathrm{ch}(J_1' + J_2' + J_3' - J_4') \\ w_7' &= 2\,\mathrm{ch}(J_1' + J_2' - J_3' + J_4'), & w_8' &= 2\,\mathrm{ch}(-J_1' + J_2' + J_3' + J_4') \end{aligned} \tag{6.1}$$

which is the same as Eqs. (3.4) with $P = 1$, and with $\hat{J}_i$ replaced by J_i'. We need the relationship between the partition functions of the two models. We assume periodic boundary conditions for the eight-vertex model, but in the present case this is not equivalent to periodic boundary conditions for the Ising model (unlike Section 3). In fact, if periodic boundary conditions are applied to the Ising model, then after the vertex spins have been summed out, the remaining face spins can only represent configurations of an eight-vertex model for which there are an even number of unbroken vertical (horizontal) edges in any row (column). To get an odd number in a row (column), it is necessary to use antiperiodic boundary conditions. Thus, the partition function of the eight-vertex model is the sum of four Ising partition functions, namely

$$2Z_{8\mathrm{V}} = Z_{\mathrm{IS,PP}} + Z_{\mathrm{IS,PA}} + Z_{\mathrm{IS,AP}} + Z_{\mathrm{IS,AA}} \tag{6.2}$$

where Z (upper case) stands for the partition function of the whole lattice with weights given by (6.1) and the subscripts PP, PA, AP, and AA refer to the four choices of boundary condition on the Ising model. This differs from the conclusion of Baxter[15]; however, in the thermodynamic limit $M, N \to \infty$, we have for the partition function per site the simple result

$$z_{8V}(w_i') = z_{IS}(J_i') \tag{6.3}$$

The left-hand side may also be evaluated from Section 3, since the eight-vertex model w_i' is equivalent under Eq. (3.20) to an Ising model with bonds $\hat{J}_i'$, while the normalization factor $\rho_{8V}(w_i')$ is readily calculated as

$$\rho_{8V}(w_i') = (w_5' w_6' w_7' w_8' - w_1' w_2' w_3' w_4')^{-1/4} = \rho_{IS}(J_i') \tag{6.4}$$

Duality relations for the eight-vertex and free-fermion models follow immediately from these results. The relations assume a simple form when we work with a nonsymmetric eight-vertex model, constructed from a given symmetric model by a similarity transformation using $H_a = H_3 = H'$ and $H_q = H_4 = H$ from Eqs. (3.22). Here we denote the weights of this nonsymmetric model by w_i. Given such an eight-vertex model, we have Ising models on the dual pair of lattices of Fig. 10. We denote a set of bonds that come from our construction of Section 3 by J_i; for each set Baxter's construction gives a second Ising model on the dual lattice with the bonds $\hat{J}_i$. Alternatively, we may use the same Ising bonds J_i on the Ising lattice and its dual, to obtain two different eight-vertex models that are dual. The first has weights u_i, given by Eqs. (3.2), from which the standard weights w_i are found using Eqs. (2.8). The second has weights $\hat{w}_i$ given by (6.1) using the same J_i. Elementary calculations show that the duality relation for the eight-vertex model is the linear transformation

$$\hat{w}_i = u_i, \qquad \hat{w}_{i+4} = \operatorname{ch} 2J_2\, u_i - \operatorname{sh} 2J_2\, u_{i+4} \tag{6.5}$$

for $i = 1, 2, 3, 4$, where u_i and w_i are related by Eqs. (2.8). This is an involution, since the inverse transformation is

$$w_i = \hat{u}_i, \qquad w_{i+4} = \operatorname{ch} 2\hat{J}_2 \hat{u}_i - \operatorname{sh} 2\hat{J}_2 \hat{u}_{i+4} \tag{6.6}$$

For the first four weights the relation assumes the same simple form regardless of the choice of symmetry for the weights, but it is more complicated for the other four with any other choice. It is obvious from Eqs. (3.18) and (4.2) that the constants Ω and h_i transform as

$$\hat{\Omega} = \Omega^{-1}, \qquad \hat{h}_i = h_i^{-1} \qquad (6.7)\,(6.8)$$

in agreement with the fact that the critical point of the model, $\Omega = 1$, $h_i = \pm 1$, $i = 2, 3, 4$, is also self-dual. The duality relations for the partition functions may now be read off as

$$\rho_{8V}(\hat{w}_1,..., \hat{w}_8)\, z_{8V}(\hat{w}_1,..., \hat{w}_8) = \rho_{8V}(w_1,..., w_8)\, z_{8V}(w_1,..., w_8) \tag{6.9}$$

and

$$\rho_{IS}(\hat{J}_i)\, z_{IS}(\hat{J}_i) = \rho_{IS}(J_i)\, z_{IS}(J_i) \tag{6.10}$$

7. CONCLUDING COMMENTS

We have introduced a formalism for two state vertex and spin models in terms of which equivalences are completely transparent, being different representations of the same algebraic objects. Because the formalism deals directly with edge variables, it is likely to be more straightforward for many purposes, and will certainly be easier to use for finite lattices. It is particularly suited for use with the quantum inverse method, and was in fact motivated as the initial step in an ongoing investigation of the relationship between the QIM and the Jordan–Wigner transformation. The QIM consists of a direct transformation to a set of operators that create Jost functions, Bethe eigenstates, etc., and an inverse transformation that recreates the original quantum fields. Local transition operators and Yang–Baxter relations are the important ingredients in the direct transform, and they have been the main subject of this paper. The inverse transform, which is usually called the quantum Gelfand–Levitan equation, may be conveniently constructed by a method given by Smirnov for a lattice version of the nonlinear Schrödinger equation.[21] The method depends on inverting the local transition operators and its application to the free-fermion model will be the subject of a future paper. However, we think it interesting to give the inversion relation here, particularly since it is obviously related to inversion for transfer matrices discussed by Baxter.[9,15] First we define the product $\mathbf{L}''$ of two local transition operators $\mathbf{L}'$, $\mathbf{L}$ as follows:

$$\mathbf{L}'' = \mathbf{L}'\mathbf{L} \tag{7.1}$$

is equivalent to

$$w''_{ik}(\alpha, \gamma) = w'_{ij}(\alpha, \beta)\, w_{jk}(\beta, \gamma) \tag{7.2}$$

where the summation convention is assumed, and the weights are matrix elements of the local transition operators as in Section 2. All three operators act at the same site. Then, for a given $\mathbf{L}$, we want to find the operator $\mathbf{L}'$ so that $\mathbf{L}''$ is the unit operator, that is, $w''_{ij}(\alpha, \beta) = \delta_{ij}\delta_{\alpha\beta}$. When Eq. (2.7) is substituted into both sides of (7.2), each of the weights u''_i is the sum of eight products of weights $u'_j u_k$, and the condition that $\mathbf{L}''$ be the unit operator is that $u''_1 = 2$, while $u''_i = 0$ for $i = 2,\ldots, 8$. For an Ising model, these products may be expressed as hyperbolic functions in the bonds J'_i and J_i using Eqs. (3.2), and when this is done the inverse turns out to be

$$\begin{aligned} &\mathbf{L}^{-1}(J_1, J_2, J_3, J_4) \\ &= (-4 \operatorname{sh} 2J_2 \operatorname{sh} 2J_4)^{-1}\, \mathbf{L}(-J_3, J_4 + i\pi/2, -J_1, J_2 + i\pi/2) \end{aligned} \tag{7.4}$$

The similarity with Baxter's work is apparent, in that inverses are obtained by a combination of reversing the signs of J_i and adding multiples of $i\pi/2$.

In conclusion, we note that the star–triangle relation is completely ubiquitous in the free-fermion and checkerboard Ising models, even in the language of the QIM, since it generates different representations of the QIM operators. Even the Yang–Baxter relations, which have been constructed using a sequence of star–triangle relations, turn out to be a single generalized star–triangle relation. It seems most likely that similar methods will be useful for elucidating relations between q-state models, such as the nonintersecting string (NIS) model investigated by Perk and Wu.(22) These questions are under investigation, and will be reported at a later time.

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